Modeling Surfaces of Arbitrary Topology
Using Complex Manifolds

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Manifolds have been used recently in computer graphics to represent two-dimensional surfaces with three-dimensional imbeddings \cite{GH95}\cite{Gri96}. We present a useful method for defining transition functions on manifolds which results from the observation that a surface locally like $\mathbb{R}^2$ is also locally like $\mathbb{C}$. We show that a constructive manifold as defined in \cite{Gri96} with charts in $\mathbb{C}$ is a differentiable manifold, and provide a method for creating $C^\infty$ transition functions on such a structure. The resulting object is easier to construct and contains fewer charts than the real manifold constructed by Grimm. Finally, we present a method for building basis functions on complex charts in order to imbed a complex manifold into $\mathbb{R}^3$. 
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1 Introduction

1.1 Overview

The differentiable manifold is a mathematical structure that defines the topology of a space in a manner that permits the space to be analyzed with the traditional tools of calculus. Manifolds can be used to describe complicated spaces of any dimension, but recently two-dimensional manifolds (also known as 2-manifolds) have been used in computer graphics to model surfaces.

An informal understanding of manifolds is most easily gained by considering the classic example of a world atlas. An atlas contains many flat rectangular pages with maps (also known as charts) of different sections of the earth. Observe that the flat charts nevertheless depict curved sections of the globe, but this does not prevent proper navigation (provided the charts represent a sufficiently small area such that the distortion is minimal). In addition, every point on the globe is represented on at least one chart, while some points appear in multiple charts. The latter condition allows you to navigate from a point on one chart to a point on another by providing continuity in the form of common areas of overlap. Note that the distortion of locations in one chart induced by flattening will in general be different than the distortion of those same points on another chart, but again, this does not prevent navigation within and among the charts.

![Figure 1: The same points on different atlas pages](image)

If the atlas charts were printed on flexible material that could stretch and warp to allow you to align the overlapping areas in common and glue them together, then you would
be able to re-construct a surface topologically like the earth. Note, however, that the resulting construction would not regain the shape of the original globe; in order to “inflate” the construction to approximate the spherical shape of the earth you would need to add information about the geometry of the globe. You could do this by specifying the locations in space for a distributed set of points on the globe. The more points specified, the more accurate the model, but a careful interpolation of a small subset of those points may provide sufficient accuracy.

The process just described is similar to the process of constructing a 2-manifold for modeling an arbitrary two-dimensional surface. In the latter case, however, we start with a polyhedral sketch instead of a surface, and we construct a 2-manifold from that sketch. To create the final surface we use the sketch to help determine how to apply geometry to the manifold.

A more extensive introduction to modeling surfaces in computer graphics with 2-manifolds can be found in [Gri96]. A clear introduction to differentiable manifolds in general can be found in [War83].

1.2 Previous work

Subdivision surfaces have been used for some time in computer graphics [CC78][Loo87]. With these techniques, sketch polyhedra are subdivided repeatedly to approximate spline surfaces. Implementation is straightforward, but the limit surface of such a procedure is only $G^1$ continuous, and not easily parameterized for operations such as texture mapping.

A different approach involves spline patches, which may be assembled in various ways to model surfaces. The resulting models have a compact representation and provide a base for useful manipulation techniques [FS94], but constraints must be maintained at patch boundaries to preserve continuity.

Other modeling techniques include constructive solid geometry (CSG) [FDFH93], and implicit surfaces [Mur91][BS91]. A CSG model is built by performing Boolean operations on geometric primitives such as spheres and cones; it quickly becomes intractable to build arbitrary surfaces using this method. Implicit surfaces are described by functions; for example the equation $x^2 + y^2 + z^2 = 1$ defines the implicit surface that is the unit sphere. Such representations are extremely efficient, but lack the malleability required for interactive modeling and do not admit a multiresolution approach.
Manifolds were recently introduced to the computer graphics field as a new way to model surfaces [GH95][Gri96]; this use of manifolds has several advantages over previous modeling techniques. Manifolds provide a compact representation of a model in a manner similar to subdivision surfaces, but without the concomitant continuity and parameterization restrictions. Manifold models also enjoy the efficient representation and local control that spline patches provide, but without the burden of maintaining constraints between portions of the surface in order to preserve continuity. Manifold models also admit parameterizations that avoid singularities such as those encountered when attempting to texture-map a sphere along longitude and latitude lines. In addition, manifolds admit a multiresolution approach; arbitrary detail can be added to a model without requiring the entire model to be represented at the finest level of detail.

1.3 Motivation

The complex manifolds constructed in this thesis offer several advantages over the real manifolds constructed by Grimm [GH95][Gri96]. These include less pre-processing, less space, and a more efficient construction. Recent approaches with real manifolds require the dual of the second subdivision of the sketch polyhedron. In contrast, complex manifolds require only that the sketch polyhedron be triangulated; an additional advantage of this is that sketch polyhedra vertices may be of any degree. Because complex manifold charts are built only on vertices of the original sketch polyhedron rather than on every element of the dual of its second subdivision as in the case of the real manifolds constructed by Grimm, complex manifolds require less space to model the same object. Furthermore, the transition functions for a complex manifold are easier to construct than those for its associated real manifold (see chapter 3).

In this thesis, we describe the explicit construction of a complex manifold from a polyhedron in 3-space, and then the construction of basis functions on the manifold from which we can build immersions. By using the vertices of the original polyhedron as control points, we can construct immersions of the underlying real manifold which closely approximate the shape of the polyhedron. From a computer graphics standpoint, this allows us to model shapes of arbitrary topology in 3-space with an infinitely-differentiable domain. Our results are simpler objects than those in [GH95] and [Gri96] in the sense that our manifolds have fewer charts for a given polyhedral sketch.
To present this construction, we first describe complex manifolds in some detail, and review the construction of manifolds from proto-manifolds that was described in [GH95] and [Gri96]; we then describe how this was adapted to produce complex manifolds instead.

1.4 Chapter summaries

The overview of this chapter provided a brief informal description of 2-manifolds in general; chapter 2 provides a more detailed informal description of complex manifolds in particular. Chapter 3 provides a more formal treatment of complex manifolds to supplement the formal discussion of real manifolds presented in [Gri96], and proves certain complex manifold properties of which we must be certain before we can attempt to use complex manifolds to model surfaces. Because chapter 3 concerns complex manifolds without geometry, chapter 4 discusses the process of immersing complex 2-manifolds in $\mathbb{R}^3$ by applying geometry through the means of basis functions. Chapter 5 presents some results of this research, and chapter 6 discusses areas for future research concerning complex manifolds. Finally, several definitions are included for convenience in the appendix.
2 Informal description of the complex manifold

2.1 Overview
Recall from section 1.1 that a 2-manifold is composed of multiple charts which are locally planar (or more precisely, locally like $\mathbb{R}^2$ for a real 2-manifold). In addition, a manifold has transition functions which determine how a point in one chart’s coordinate system corresponds to a point in the coordinate system of another chart. The formal definition of a manifold includes additional conditions, such as the cocycle condition, which requires that the transition from the point $x$ on chart $c_i$ to chart $c_j$ to chart $c_k$ is the same as the transition of $x$ from $c_i$ directly to $c_k$ (see Figure 2).

![Figure 2: Charts, overlap regions, and the cocycle condition](image)

Note that the transition functions and the cocycle condition are analogous to the warping and gluing of the atlas charts described in section 1.1.

One question that naturally arises is whether the transition functions must map through $\mathbb{R}^2$, or if the complex plane could be used, instead. Intuitively it seems that the topology of the chart is independent of its interpretation and that if the charts were lying in $\mathbb{C}$ rather than $\mathbb{R}^2$, the atlas would still define a manifold. In addition, it seems clear that the addition of the standard mappings from $\mathbb{R}^2$ to $\mathbb{C}$ and back would not change the relationships between the points found on the charts. Finally, it seems
natural that a transition function mapping a point from one Cartesian coordinate system to another can just as easily map from one complex coordinate system to another. These observations lead to an efficient transition function that takes advantage of a property of the complex number space which proves useful when applied locally to portions of the polyhedral sketch.

2.2 The polyhedral sketch

The polyhedral sketch is the scaffolding upon which the manifold is built; it is a rough sketch of the shape of the desired surface. In our case it must be triangulated (for reasons that become apparent in section 2.4), orientable (see section 4.1), and contain no boundaries (see chapter 6). However, it may contain vertices of any degree. The complex manifold is built on top of the polyhedral sketch. We begin by associating a chart with a neighborhood of each vertex of the sketch polyhedron. We then define transition functions between the charts, and finally we place basis functions on top of the charts in order to define what shape the manifold will take.

2.3 The vertex charts

In order to associate a chart with a neighborhood of a vertex in a manner that allows us to map that portion of the manifold to something locally like $\mathbb{R}^2$ or $\mathbb{C}$, we must find a way to flatten the area immediately surrounding the vertex. To do this we cut along one edge leading away from the vertex and then along each far edge of every face in the star of that vertex. We then orient that flattened portion of the polyhedron in $\mathbb{C}$ such that the vertex is at the origin and the first edge we cut is along the positive real axis, as shown in Figure 3.
Note that the flattening does not define a continuous map from the neighborhood of the vertex to $\mathbb{R}^2$ (or $\mathbb{C}$), but will be used as an auxiliary first step in the construction of the actual chart.

### 2.4 The transition functions

Once we have flattened a portion of the polyhedron in the manner described above, we can apply a useful aspect of the complex plane in order to create a chart in the manifold. Observe that a complex number $z=r e^{i\theta}$ raised to the exponent $n$ has the effect of multiplying the argument of the number by $n$ (recall that $z^n=r^n e^{i\theta n}$). Although an exponent changes the radius of $z$, it also rotates $z$ around the complex origin. Notice, for example, that a set of complex numbers $W$ spread throughout the upper complex half-plane...
plane all raised to the power of 2 would be mapped to the set of points \( W^2 = \{z^2 \mid z \in W\} \) spread throughout the entire complex plane.

We take advantage of this effect and warp the unfolded polyhedral section in the complex plane so that the faces in the vertex star that were split along the first edge are rejoined. To do this, we first determine what fraction \( f \) of a circle the flattened face star spans\(^1\). We then map the plane to itself with the map \( z \rightarrow z^{1/f} \). This result, composed with our initial map, is actually a continuous one-to-one map from the vertex star to a neighborhood of the origin in the complex plane. The resulting surface lies entirely within \( \mathbb{C} \), is star-shaped with respect to the origin, and provides an easy method for defining transition functions between charts.

![Figure 4: Warp of a face star](image)

We can use this technique to transition a point from one chart to another on the manifold by applying in succession a warp, a rotation, a translation, a rotation, and an unwarp. These operations can be composed into one equation:

\[
z_b = e^{i\theta_b} \left( \frac{a - b}{\|a - b\|} - (e^{-i\theta_a} z_a)^{(f_i h)} \right)^{(1/f_i h)}
\]

where:

1) \( z_a \) represents any point in the chart associated with a neighborhood of vertex \( a \), while \( z_b \) is that same point expressed in the coordinate system for the chart associated with vertex \( b \);

2) \( f_i \) is the fraction of a circle swept by the face star of vertex \( i \);

\(^1\) Note that for saddle points this fraction can be greater than 1.
3) $e^{i \theta_i}$ is the multiplier which rotates a complex point by $\theta_{ij}$ which is the angle at which vertex $j$ appears in the chart for vertex $i$.

4) $z^{[n]w}$ is the $n$th power of $z$ which is closest to the point $w$; we call this the $n$th $w$-oriented power of $z$. (See Figure 5).

![Figure 5: A 1-oriented power applied to a chart](image)

The transition function defined above first rotates the point $z_a$ so that point $b$ as it appears in chart $a$ lies along the real axis. The 1-oriented power is taken by the amount which will unwarp chart $a$, and then the point is translated along the real axis by the distance between the two chart vertices so that vertex $b$ is at the origin. The point is then warped by the amount necessary to close chart $b$, and then rotated so that point $a$ appears in the correct location in chart $b$.

### 2.5 Orientable Manifolds

One technical detail concerning manifolds used for modeling is that such manifolds must be orientable [War83]. The definition is highly technical, but may be intuitively understood by considering two counter-examples: neither the Möbius strip nor the Klein bottle is an orientable manifold. The problem with trying to represent such surfaces with manifolds arises when a local basis is chosen. When the local basis is translated along the surface it may appear again at the original location in the manifold with a different orientation, which indicates that the cocycle condition may fail.
3 Formal description of the complex manifold

Manifolds have been used since the late 1800's and early 1900's in mathematics, and were introduced to computer graphics in [GH95] and [Gri96]. The fundamental difference between the way the two fields treat manifolds is that in mathematics a manifold is built on top of an existing surface, and in computer graphics a manifold is constructed in order to create a surface. For a detailed formal introduction to traditional and constructive manifolds, the reader is referred to [Gri96].

3.1 Proof that M^K with complex charts is a manifold

We begin by presenting a formal definition of a constructive complex manifold adapted from [Gri96]:

**Definition 1** A $C^k$-differentiable complex proto-manifold $K$ of dimension $2n$ consists of the following:

1. A finite set $A$ of open sets in $\mathbb{C}^n$. $A$ is called the proto-atlas. Each element $c \in A$ is called a chart.

2. The subset $U_{cc'} \subset c$ for every pair of charts $c, c' \in A$. If $c = c'$ then $U_{cc'} = c$.

3. A set of functions $\Phi$, called transition functions. For every pair of charts $c, c' \in A$ the transition function $\varphi_{cc'} \in \Phi$ is a map $\varphi_{cc'} : U_{cc'} \rightarrow U_{c'c}$. Note that $U_{cc'}$ and $U_{c'c}$ may be empty. The following conditions on the transition functions must hold:

   - (a) $\varphi_{cc'}$ is one-to-one, onto, and $C^k$-differentiable
   - (b) $\varphi_{cc'}^{-1} = \varphi_{c'c}$
   - (c) $\forall x \in U_{cc}$ it must be the case that $\varphi_{cc}(x) = x$
   - (d) The cocycle condition: $(\varphi_{ik} \circ \varphi_{jk})(x) = \varphi_{ik}(\varphi_{jk}(x))$ for $x \in U_{ik} \cap U_{jk}$

4. An equivalence relation, defined in the following manner. Given two charts $c$ and $c'$ in $A$, define a relation on the disjoint union of these charts, $c \cup c'$, as follows: if $x \in c$, $y \in c'$, and $y = \varphi_{cc}(x)$ then $x \sim y$. 
Note that if we let $K_{cc'}$ be the quotient of $c \sqcup c'$ by $\sim$ as defined in element 4 of the definition, then there must exist an embedding $\mathcal{E}_{cc'}$ of $K_{cc'}$ into $\mathbb{C}^n$. The purpose of this condition is to ensure that $M^K$ is a Hausdorff space. Note also that complex proto-manifolds may only be of real dimension $2n$, in contrast with real proto-manifolds, which may be of any dimension greater than zero.

We're now ready to define the constructive complex manifold; this definition is from [GH95]:

**Definition 2** Let $\sim$ be the equivalence relation described above and $K$ a complex proto-manifold as defined above. Define $M^K$ as the quotient of $\sqcup_{c \in A}^c c$ by $\sim$. Let $\Pi$ be the map taking $x \in \sqcup_{c \in A} c$ to $[x] \in M$, where $[x]$ is the equivalence class of $x$.

The proof that $M^K$ is a real manifold follows easily from the fact that we can provide the obvious mapping from $\mathbb{C}^n$ to $\mathbb{R}^m$, where $m = 2n$. The proof that $M^K$ with maps to $\mathbb{R}^m$ is a manifold is provided in [Gri96]. To prove that $M^K$ is a complex manifold, we must establish differentiability of the transition functions. This is a subject of the following section.

### 3.2 Proof that the constructive complex manifold is $C^k$ for a given $k$

Note that, except along the negative real axis where the 1-oriented power splits a chart (see Figure 5), the transition function is continuous because it is a combination of continuous functions. Consider charts $a$ and $b$ from the transition function formula in section 2.4. The face star of chart $a$ intersected with the face star of chart $b$ is the overlap of the two charts in the manifold itself, and therefore the star for each chart maps into a region in the coordinate space of each adjacent chart that doesn’t intersect the negative real axis. The only possible situation where this would not be the case *would* be if the angle swept by one face of a vertex star was greater than $\pi/2$, while the sum of the angles swept by the remaining faces was less than $\pi/2$. In that case, warping the face star would cause a shared face to cross the negative real axis, but note that this is not possible in a polyhedron without boundary; the angle swept by one face in a star cannot exceed the sum of the angles swept by the remaining faces. Therefore the continuity of the surface generated by an immersed complex manifold is limited only by the choice of basis functions, and not by the transition functions.
4 Immersing the complex manifold

4.1 Overview

Recall that a manifold contains only topological information, and that geometry must be applied to a 2-manifold before it can be used as a surface model. Such geometrical information is usually described by a mapping from the object into 3- (or n-) dimensional space by a mapping that has the following properties. When composed with a coordinate chart map from $\mathbb{R}^k$ to $M$ (generating a map that goes from $\mathbb{R}^k$ to $M$ to $\mathbb{R}^n$) the mapping must be one-to-one and differentiable at every point of every chart, and the derivative is required to have rank $k$ at every point. We will define such maps via *basis functions*. Combinations of basis functions will not always be immersions, but generally they will.

Basis functions by definition must be non-zero somewhere, must overlap, must be $C^k$ for a given $k$, and must have limited support. These requirements help constrain basis function endpoint derivatives and values. In computer graphics basis functions are most commonly encountered when constructing splines; see [Gri96] for a more detailed discussion of traditional basis functions.

4.2 Intractable basis functions

Several obvious approaches to placing basis functions on complex manifolds prove intractable in practice. These include b-spline basis functions and closed-form solutions.

4.2.1 B-spline basis functions

The techniques for modeling with real manifolds described in [GH95] and [Gri96] use b-spline basis functions laid out for the most part in rectangular knot grids. Unfortunately, the warp and unwarp components of the complex manifold transition functions would cause those basis functions to be distributed non-uniformly, as shown in Figure 6.
4.2.2 Closed-form complex solutions

We could attempt to solve for the analytic complex-valued solution to the problem of placing a single basis function on a chart. However, a remarkable result in complex analysis that stems from the Cauchy integral formula shows us that this is not possible:

**Liouville’s Theorem**: If $f$ is entire and bounded for all values of $z$ in the complex plane, then $f(z)$ is constant. [CBV74]

By definition our basis functions are entire (analytic throughout the complex plane) and bounded (finite), and therefore the only basis functions satisfying these requirements are constant. Thus the only closed-form analytic solution also satisfying our boundary conditions is the constant mapping $f(z) = 0$, which will certainly not suffice.

4.2.3 Closed-form real-valued solutions

The above constraints suggest an additional possibility, which would be to solve directly for the real-valued solution for placing a single basis function on a chart. This is reasonable because we are interested in merely associating some real value for every point in the complex plane for each chart. However, the multiple boundary conditions generate intractably difficult problems, the solutions of which can not be easily represented algorithmically. A symbolic solver could be incorporated to find closed-form real-valued solutions, but the running time would not allow interactive modeling.
4.3 The basis functions used in this research

We avoid problems associated with the non-uniform distribution of knot grids by building a basis function on each chart. By associating a chart with each vertex, we provide sufficient chart overlap (two along original edges and three on the interior of each face) to guarantee that the image of the immersion is always a parameterization of the original polyhedron. However, in order to provide sufficient continuity of the surface such that changing control points will result in useful changes to the surface, additional charts must be added; this is the subject of future work (see chapter 6).

4.3.1 A product of functions

Having abandoned the hopeless problem of finding analytic basis functions, we instead construct $C^k$ real-valued basis functions. To do so, we take advantage of the fact that the product of several $C^k$ functions is $C^k$. To build our basis functions, we essentially multiply one function per face in the star of the chart’s vertex.

Ideally, the formula for the basis function for a chart would be

$$b(z) = \prod_i g_i(z)$$

where the product is taken over the $i$ faces of the star of the chart’s vertex. However, this would not preserve continuity along the first edge that we cut (see Figure 7).

![Figure 7: Discontinuity due to differing calculations](#)

In order to preserve continuity we abut an additional virtual copy of the face star along each side of that edge when evaluating the basis function for the associated chart (see Figure 8). This step is repeated as necessary to assure that when the basis function is calculated on any face in the original star, the result includes contributions from all face functions that could possibly affect the value at that point.
4.3.2 The face function

We're now ready to define $g_i(z)$, which should equal one at the origin and fade to zero at the far edge of face $i$:

$$g_i(z) = f\left(\frac{\langle m_i - o_i, z - o_i \rangle}{\|m_i - o_i\|}\right)$$

Where $o_i$ is the origin of chart $i$, and $m_i$ is the point closest to $o_i$ along the edge opposite $o_i$, $\langle a, b \rangle$ denotes the inner product of the vectors $a$ and $b$, and

$$f(t) = \begin{cases} 
1 & t \leq 0 \\
4t^3 - 4t^2 + 1 & 0 < t < 1/2 \\
4t^3 - 8t^2 + 4t & 1/2 < t < 1 \\
0 & t \geq 1 
\end{cases}$$

The function $f$ was constructed to be $C^3$ and satisfy the following constraints:

$$f(0) = 1, \quad f'(0) = 0, \quad f'(1/2) = 1/2, \quad f''(1/2) = -1, \quad f(1) = 0, \quad f'(1) = 0$$

Figure 9 depicts a plot of $f$ on the domain $[0, 1]$. 

**Figure 8: Virtual face star copies**
$m_i$ is found by taking the projection of an edge $(o_i, v_i)$ in the star of $o_i$ onto the edge opposite $o_i$ and then adding that projection to $v_i$.

4.3.3 The partition of unity

Note that the support of the basis functions described above are not guaranteed to be evenly distributed throughout the manifold, and that there are likely to be points on the manifold that have insufficient basis function support. If the basis functions do not sum to one at a point, then the manipulation of points around that region will have an undesired non-uniform effect. One solution is to force the basis functions to always sum to one by applying a normalization; this is known as a partition of unity [War83], and is usually accomplished by dividing the original result by the sum of the basis functions. Note that this is only possible if the sum is guaranteed to be non-zero. This condition should be satisfied by the facts that every point on the complex manifold should lie within the interior of at least one chart, and that the basis functions for any chart will be non-zero everywhere within the boundary of that chart.
5 Research results

Figure 10 provides a visual demonstration of the continuity of the transition functions for a complex manifold built on top of a sketch polyhedron. It is apparent that the arrow drawn in the star of vertex 3 is mapped conformally to the adjoining charts and their associated face stars.
Figure 11 shows a sketch polyhedron and a low-resolution tesselation of its associated complex manifold.
6 Future Work

The next step required to make complex manifolds useful for modeling surfaces is to provide additional charts in order to guarantee sufficient basis function overlap such that modifying control points results in useful changes to the surface. One way to approach this might be to add an additional chart centered at each face of the original sketch polyhedron. However, to preserve continuity at the original vertex points with this approach, the control points for the surrounding faces' basis functions would need to be constrained to lie in a plane, thus artificially restricting the possible modifications to the model.

The implementation described herein is restricted to triangulated sketch polyhedra; it should be possible to extend the construction of the transition functions to include those vertices which share a face but not an edge. However, note that this will require a new approach for the formulation of the transition and basis functions.

It should also be possible to extend this research to include manifolds with boundary. One problem that must be dealt with in this case is the fact that the basis functions centered at boundary vertices will be undefined beyond the boundary, and a solution must therefore be carefully chosen to preserve the continuity of those functions.

The problem with non-orientable manifolds mentioned in section 2.5 could possibly be addressed in future research. This may be soluble simply by restricting the construction of manifolds to those cases where it can be guaranteed that the surface will be locally orientable so that the cocycle condition holds among charts for which transition functions are defined.

Some additional research topics mentioned in [GH95] and [Gri96] apply to complex manifolds as well; these include introducing discontinuities into surfaces, immersing curves into complex manifolds, and texture mapping in parameter space.
Appendix: Definitions

The following definition of Hausdorff space is from [Bau64], p. 40. In the definition, $\mathcal{L}_x$ is the system of neighborhoods at $x$. A topological space must be Hausdorff to permit sequence limits, a necessary condition for differentiation.

**Definition 3** Let $(X, \mathcal{J})$ be a topological space, then $\mathcal{J}$ is said to be a Hausdorff (or $T_2$) topology for $X$, provided that for each pair, $x, y,$ with $x \neq y,$ of points of $X$, there exist $U \in \mathcal{L}_x$ and $V \in \mathcal{L}_y$ such that $U \cap V = \emptyset$. If this condition is satisfied we call $X$ a Hausdorff (or $T_2$) space.

The following definitions of immersion and imbedding are from [War83], p. 22. In these definitions, the map $\psi: M \to N$ is $C^\infty$:

**Definition 4** $\psi$ is an immersion if $d\psi_m$ is non-singular for each $m \in M$.

**Definition 5** $\psi$ is an imbedding if $\psi$ is a one-to-one immersion which is also a homeomorphism into; that is, $\psi$ is open as a map into $\psi(M)$ with the relative topology.

The following definition of orientable is from [War83], p. 139.

**Definition 6** Let $M$ be a differentiable manifold of dimension $n$. Then the following are equivalent:

(a) $M$ is orientable.

(b) There is a collection $\Phi = \{(V, \psi_i)\}$ of coordinate systems on $M$ such that

$$M = \bigcup_{(V, \psi_i) \in \Phi} V$$

and

$$\det \begin{vmatrix} \frac{\partial x_i}{\partial y_j} \end{vmatrix} > 0 \quad \text{on} \quad U \cap V$$

whenever $(U, x_1, \ldots, x_n)$ and $(V, y_1, \ldots, y_n)$ belong to $\Phi$.

(c) There is a nowhere-vanishing $n$-form on $M$. 

\[20\]
Bibliography


